

Symmetric mixed states of n qubits: local unitary stabilizers and entanglement classes

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We classify, up to local unitary equivalence, local unitary stabilizer Lie algebras for symmetric mixed states of n qubits into six classes. These include the stabilizer types of the Werner states, the GHZ state and its generalizations, and Dicke states. For all but the zero algebra, we classify entanglement types (local unitary equivalence classes) of symmetric mixed states that have those stabilizers. We make use of the identification of symmetric density matrices with polynomials in three variables with real coefficients and apply the representation theory of $SO(3)$ on this space of polynomials.

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I. INTRODUCTION

Quantum information seeks to exploit quantum states as resources for computational tasks, where the phenomenon of entanglement seems to play a central role in quantum advantages over classical protocols. A problem at the heart of the study of entanglement is to understand the essential differences between local and global unitary operations on composite systems. Questions such as the following arise naturally. How can one characterize the nonlocal properties of a given state? Given an input state, what is a reasonable description of the set of states to which it can be converted using only local operations? These questions are known to be hard. Nonlocal properties are poorly understood [1]. Regarding the second question, the number of polynomial invariants needed to determine local unitary equivalence classes of multiqubit states grows exponentially with the number of qubits [2].

The considerations of the previous paragraph motivate the search for entanglement measures and local unitary invariants that, being more tractable than a complete set of polynomial invariants, will provide a coarser grained classification, yet still fine enough to distinguish useful states and good entanglement properties. We pursue this philosophy with the following scheme. Given an n -qubit density matrix ρ , its local unitary stabilizer subgroup is defined to be all the local transformations of state space that leave ρ invariant. If ρ, ρ' are local unitary equivalent, say by a local unitary operation U , then their stabilizers are isomorphic via conjugation by U [3]. Thus the conjugacy class of the stabilizer subgroup of a state is a local unitary invariant of that state. This invariant is practical and has proven to distinguish states with known useful entanglement properties. Tractability is achieved by virtue of the Lie group structure of the local unitary group and stabilizer subgroups. Lie groups admit analysis through their Lie algebras, which are their tangent spaces of infinitesimal local operations. In previous

work, we have classified conjugacy classes of stabilizers and local unitary classes of states including the singlet, GHZ and its generalizations, symmetric Dicke states (including the W state and its generalizations), and Werner states [4–8].

In this article, we consider the class of symmetric mixed states of n -qubit systems. Symmetric states, that is, states that are invariant under permutation of subsystems, are the subject of recent work including: geometric measure of entanglement [9–13], efficient tomography [14], classification of states equivalent under stochastic local operations and classical communication (SLOCC) [15–17], and our own work on classification of pure symmetric states equivalent under local unitary (LU) transformations [3, 7].

The main results of this paper are: a classification of the six local unitary stabilizer subalgebras (Lie algebras of the local unitary stabilizer subgroups) in Theorem 2; and for each of those algebras (except for the zero algebra), a classification of local unitarily distinct classes of states. In addition, Theorem 1 gives a structure theorem for stabilizer subalgebras of mixed states that generalizes a similar result for pure states in our earlier work [4].

The paper is organized as follows. We establish notation and convention in Section II. Section III presents the identification of symmetric mixed states with polynomials in 3 variables and the basics of $SO(3)$ representation theory (already given in our previous work [8] but reproduced here for the sake of self containment) needed for the analysis of stabilizers and their corresponding states. We give structure theorems for subalgebras of the local unitary algebra in general (Theorem 1), and stabilizer algebras of symmetric mixed states in particular (Theorem 2), in Section IV. Proofs of these theorems are given in the appendix. Five subsequent sections analyze and classify entanglement types of symmetric mixed states (local unitary equivalence classes) for five of the six stabilizer types (all but the zero algebra) given in Theorem 2.

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II. PRELIMINARIES

Let $G = (SU(2))^n$ denote the n -qubit local unitary group, where $g = (g_1, g_2, \dots, g_n) \in G$ acts on a density matrix ρ by

$$g\rho g^\dagger := (g_1 \otimes g_2 \otimes \dots \otimes g_n)\rho(g_1 \otimes g_2 \otimes \dots \otimes g_n)^\dagger.$$

Let $LG = \bigoplus_{i=1}^n su(2)$ denote the local unitary Lie algebra of infinitesimal transformations, where $su(2)$ is the set of 2×2 skew-Hermitian matrices with trace zero. In order to study the actions of G and LG on density matrices, it is convenient to consider a larger vector space that contains the set of density matrices as a proper subset. Let \mathcal{W} denote the 4-dimensional real vector space of 2×2 Hermitian matrices. A convenient basis for \mathcal{W} is $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, where σ_0 is the 2×2 identity matrix, and $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, and $\sigma_3 = \sigma_z$ are the Pauli matrices.

The set of n -qubit density matrices is a proper subset of the vector space $\mathcal{W}^{\otimes n}$, where every element ρ (whether or not ρ is positive or has trace 1) can be uniquely written in the form $\rho = \sum_I s_I \sigma_I$, where $I = i_1 i_2 \dots i_n$ is a multiindex with $i_k = 0, 1, 2, 3$ for $1 \leq k \leq n$, and σ_I denotes

$$\sigma_I = \sigma_{i_1} \otimes \sigma_{i_2} \otimes \dots \otimes \sigma_{i_n},$$

and the coefficients s_I are real. When it is necessary to express a density matrix in terms of the computational basis, we use the notation $\rho = \sum_{I,J} \rho_{I,J} |I\rangle \langle J|$, where $I = i_1 i_2 \dots i_n$ and $J = j_1 j_2 \dots j_n$ are bit strings of length n . For $I = i_1 i_2 \dots i_n$, we write I_k to denote the string $i_1 i_2 \dots i_k^c \dots i_n$ obtained by complementing the k th index, where $i_k^c = i_k + 1 \pmod{2}$. Note that the same multiindex notation $I = i_1 i_2 \dots i_n$ means $i_k = 0, 1, 2, 3$ for the Pauli tensor expansion, and means $i_k = 0, 1$ for the computational basis expansion, where context makes the meaning clear.

An element $g \in SU(2)$ acts [20] on σ_i by

$$\sigma_i \mapsto g \sigma_i g^\dagger \quad (1)$$

and the corresponding action of M in $su(2)$ on σ_i is

$$\sigma_i \mapsto [M, \sigma_i] = M \sigma_i - \sigma_i M. \quad (2)$$

An element $g = (g_1, g_2, \dots, g_n)$ in G acts on σ_I by

$$g \sigma_I g^\dagger := (g_1 \sigma_{i_1} g_1^\dagger) \otimes \dots \otimes (g_n \sigma_{i_n} g_n^\dagger) \quad (3)$$

and $M = (M_1, M_2, \dots, M_n)$ in LG acts on σ_I by

$$[M, \sigma_I] := \sum_{k=1}^n \sigma_{i_1} \otimes \dots \otimes [M_k, \sigma_{i_k}] \otimes \dots \otimes \sigma_{i_n} \quad (4)$$

The local unitary stabilizer of $\rho \in \mathcal{W}^{\otimes n}$ is the subgroup Stab_ρ of G given by

$$\text{Stab}_\rho = \{g \in G : g\rho g^\dagger = \rho\}.$$

The corresponding local unitary stabilizer subalgebra, which we denote by K_ρ , is the Lie algebra of Stab_ρ , given by

$$K_\rho = \{M \in LG : [M, \rho] = 0\}.$$

We will use the standard basis

$$A = i\sigma_z, \quad B = i\sigma_y, \quad C = i\sigma_x$$

for $su(2)$. Given an element $M \in su(2)$, we will write $M^{(k)}$ to denote the element $(0, 0, \dots, M, \dots, 0)$ in LG with M in the k th position and zero matrices in all other positions.

We will make use of the following elementary 1-qubit Lie algebra actions on Pauli tensors.

$$[A, \sigma_x] = -2\sigma_y \quad (5)$$

$$[A, \sigma_y] = 2\sigma_x \quad (6)$$

$$[A, \sigma_z] = 0 \quad (7)$$

Less trivial, but still elementary to check, are the following Lie algebra action calculations in standard coordinates [5].

$$\left[\left(\sum_k a_k A^{(k)} \right), \rho \right] = \sum_{I,J} \zeta(I, J) |I\rangle \langle J| \quad (8)$$

$$\left[\left(\sum_k c_k C^{(k)} \right), \rho \right] = \sum_{I,J} \eta(I, J) |I\rangle \langle J| \quad (9)$$

where $\zeta(I, J), \eta(I, J)$ are given by

$$\zeta(I, J) = 2i \rho_{I,J} \sum_{i_\ell \neq j_\ell} (-1)^{i_\ell} a_\ell$$

$$\eta(I, J) = i \sum_{k,\ell=1}^n (c_k \rho_{I_k, J} - c_\ell \rho_{I, J_\ell}).$$

III. SYMMETRIC STATES AND POLYNOMIALS

Given a permutation π of $\{1, 2, \dots, n\}$, define P_π to be the operator on Hilbert space that carries out the corresponding permutation of qubits. For example,

$$P_{(23)} |11000\rangle = |10100\rangle$$

$$P_{(23)} (\sigma_j \otimes \sigma_k \otimes \sigma_l) P_{(23)}^{-1} = \sigma_j \otimes \sigma_l \otimes \sigma_k$$

Define a symmetrization operator on density matrices as

$$\text{Sym}(\rho) = \frac{1}{n!} \sum_{\pi} P_\pi \rho P_\pi^{-1}.$$

An n -qubit symmetric density matrix ρ is one for which $P_\pi \rho P_\pi^{-1} = \rho$ for all permutations π , or equivalently, one for which $\text{Sym}(\rho) = \rho$. We denote by $\text{Sym}^n \mathcal{W}$ the n -fold

symmetric power of \mathcal{W} defined in the previous section. It is the subspace of elements of $\mathcal{W}^{\otimes n}$ that are invariant under qubit permutation. Every n -qubit symmetric density matrix ρ is an element of $\text{Sym}^n \mathcal{W}$, and can be written

$$\rho = \frac{1}{2^n} \sum c_{n_1 n_2 n_3} \text{Sym}(\sigma_0^{\otimes n_0} \otimes \sigma_1^{\otimes n_1} \otimes \sigma_2^{\otimes n_2} \otimes \sigma_3^{\otimes n_3}), \quad (10)$$

where the sum is over non-negative integers n_0, n_1, n_2 , and n_3 such that $n_0 + n_1 + n_2 + n_3 = n$. The coefficients $c_{n_1 n_2 n_3}$ are real. The collection of n -qubit symmetric density matrices is a proper subset of $\text{Sym}^n \mathcal{W}$, since the latter contains Hermitian matrices that are not positive semi-definite, and Hermitian matrices for which the trace is not 1.

Let $\mathbb{R}_n[x, y, z]$ be the set of polynomials of degree at most n in three variables x, y , and z with real coefficients. For each n , there is a linear map $F_n : \text{Sym}^n \mathcal{W} \rightarrow \mathbb{R}_n[x, y, z]$ defined by

$$\frac{1}{2^n} \text{Sym}(\sigma_0^{\otimes n_0} \otimes \sigma_1^{\otimes n_1} \otimes \sigma_2^{\otimes n_2} \otimes \sigma_3^{\otimes n_3}) \mapsto x^{n_1} y^{n_2} z^{n_3}.$$

In this way, we may associate a polynomial of degree at most n with each n -qubit symmetric mixed state. The polynomial associated with (10) is

$$F_n(\rho) = \sum_{\substack{n_1, n_2, n_3 \\ n_1 + n_2 + n_3 \leq n}} c_{n_1 n_2 n_3} x^{n_1} y^{n_2} z^{n_3}. \quad (11)$$

For each n , the map F_n is an invertible linear map.

Since F_n is a linear map, the polynomial for a mixture of symmetric mixed states is the mixture of the polynomials.

$$F_n(p_1 \rho_1 + p_2 \rho_2) = p_1 F_n(\rho_1) + p_2 F_n(\rho_2)$$

A product of polynomials $\mathbb{R}_n[x, y, z] \times \mathbb{R}_m[x, y, z] \rightarrow \mathbb{R}_{n+m}[x, y, z]$ represents the symmetrized tensor product of states.

$$F_n(\rho_1) F_m(\rho_2) = F_{n+m}(\text{Sym}(\rho_1 \otimes \rho_2))$$

Let $g \in SU(2)$. Define $T_g : \text{Sym}^n \mathcal{W} \rightarrow \text{Sym}^n \mathcal{W}$ to be the symmetric transformation of each qubit by g .

$$T_g(\rho) = g^{\otimes n} \rho (g^\dagger)^{\otimes n}$$

If we define $R_g : \mathbb{R}_n[x, y, z] \rightarrow \mathbb{R}_n[x, y, z]$ to be the transformation on polynomials defined by

$$R_g(f)(x, y, z) = f((x, y, z)\Phi(g)), \quad (12)$$

where f is a polynomial, $\Phi : SU(2) \rightarrow SO(3)$ is the homomorphism [21] that associates a rotation in \mathbb{R}^3 with each 2×2 unitary, and $(x, y, z)\Phi(g)$ denotes a row vector multiplied by a 3×3 orthogonal matrix, then the following diagram commutes.

$$\begin{array}{ccc} \mathbb{R}_n[x, y, z] & \xrightarrow{R_g} & \mathbb{R}_n[x, y, z] \\ F_n \uparrow & & \uparrow F_n \\ \text{Sym}^n \mathcal{W} & \xrightarrow{T_g} & \text{Sym}^n \mathcal{W} \end{array}$$

Homogeneous polynomials in three variables x, y , and z are known to be reducible representations of $SO(3)$ [18][22]. If V_l is the irreducible representation of $SO(3)$ with dimension $2l+1$, then the homogeneous polynomials of degree p in three variables decompose into irreducible representations as

$$\bigoplus_{j=0}^{\lfloor p/2 \rfloor} V_{p-2j}. \quad (13)$$

IV. STABILIZER STRUCTURE FOR SYMMETRIC MIXED STATES

We begin by considering an arbitrary subalgebra K (not necessarily a stabilizer subalgebra) of $LG = \bigoplus_{i=1}^n su(2)$. That is, K is a vector subspace of LG that is also closed under the Lie bracket. Theorem 1 below (the proof is given in the appendix) gives a structure theorem for how K decomposes as a direct sum of basic building blocks. It is convenient to give names to the following standard algebra types.

Given a subset $\mathcal{B} \subset \{1, 2, \dots, n\}$ of qubit labels for a system of n qubits, let $\Delta_{\mathcal{B}}$ denote the subalgebra

$$\Delta_{\mathcal{B}} = \left\{ \sum_{b \in \mathcal{B}} M^{(b)} : M \in su(2) \right\}$$

of LG . In particular, note that $\Delta_{\mathcal{B}}$ is isomorphic to $su(2)$. We will call $\Delta_{\mathcal{B}}$ the *standard $su(2)$ block subalgebra* for the subsystem specified by \mathcal{B} , and we will use the term *$su(2)$ block algebra* for any algebra that is isomorphic to some $\Delta_{\mathcal{B}}$ via local unitary group conjugation, that is, any algebra of the form

$$(g_1 \otimes g_2 \otimes \dots \otimes g_n) \Delta_{\mathcal{B}} (g_1 \otimes g_2 \otimes \dots \otimes g_n)^\dagger$$

where $g_i \in SU(2)$ for $1 \leq i \leq n$. We will write K_{Werner} to denote $\Delta_{\{1, 2, \dots, n\}} = \{(M, M, \dots, M) : M \in su(2)\}$, because it is the Lie algebra of the stabilizer group $\{(g, g, \dots, g) : g \in SU(2)\}$ that defines the Werner states.

We write K_{product} to denote the n -dimensional algebra

$$K_{\text{product}} = \left\{ \sum_{k=1}^n a_k A^{(k)} : a_k \in \mathbb{R} \right\}$$

and we write K_{Dicke} to denote the 1-dimensional algebra that is the real span of the element $\sum_{k=1}^n A^{(k)}$. We use these names because K_{product} is the stabilizer of the product state $|00 \dots 0\rangle \langle 00 \dots 0|$, and K_{Dicke} is the stabilizer of the symmetric Dicke states [7]. Finally, let K_{GHZ} denote the $(n-1)$ -dimensional algebra

$$K_{\text{GHZ}} = \left\{ \sum_{i=1}^n a_i A^{(i)} : a_i \in \mathbb{R}, \sum_i a_i = 0 \right\}$$

which is isomorphic to $u(1)^{(n-1)}$. The name of this algebra comes from the fact that it is the stabilizer of the GHZ state[5].

Theorem 1. *Let K be a subalgebra of $LG = \bigoplus_{i=1}^n su(2)$. The set $\{1, 2, \dots, n\}$ decomposes as a disjoint union*

$$\{1, 2, \dots, n\} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_p \cup \mathcal{S} \cup \mathcal{R}$$

such that the algebra K decomposes as a direct sum of algebras

$$K = \left(\bigoplus_{i=1}^p B_i \right) \oplus S$$

with the following properties:

- (i) *each B_i is an $su(2)$ block subalgebra in qubits \mathcal{B}_i ,*
- (ii) *every element in the algebra S has zero coordinates in qubits outside of the qubit set \mathcal{S} ,*
- (iii) *for every s in \mathcal{S} , the set $\{M_s : \sum_k M_k^{(k)} \in S\}$ is a 1-dimensional subalgebra of $su(2)$, and*
- (iv) *every element in K has zero coordinates in qubits in \mathcal{R} .*

Applying Theorem 1 to the stabilizer subalgebra K_ρ , where ρ is symmetric, leads to the following theorem, whose proof is in the appendix.

Theorem 2. *Let ρ be a symmetric mixed n -qubit state. There is an LU equivalent symmetric state ρ' such that the local unitary stabilizer subalgebra $K_{\rho'}$ is one of the following.*

- (a) *all of LG*
- (b) *K_{Werner}*
- (c) *K_{product}*
- (d) *K_{GHZ}*
- (e) *K_{Dicke}*
- (f) *the zero algebra*

In the following sections, we identify states that have the stabilizers given in Theorem 2. For stabilizer types (c), (d), and (e), we will make use of the following Lemma, whose proof is in the appendix.

Lemma 1. *Suppose that K_ρ is a stabilizer of type (c), (d), or (e), in Theorem 2, and that*

$$(g_1 \otimes g_2 \otimes \dots \otimes g_n) K_\rho (g_1 \otimes g_2 \otimes \dots \otimes g_n)^\dagger = K_\rho$$

for some g_1, \dots, g_n in $SU(2)$. Then we have $g_k A g_k^\dagger = \pm A$ for $1 \leq k \leq n$. It follows that each g_k is either diagonal or antidiagonal. That is, g_k is either of the form $g_k = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$ or $g_k = \begin{bmatrix} 0 & -e^{-it} \\ e^{it} & 0 \end{bmatrix}$ for some real t .

V. STABILIZER IS ALL OF LG

The completely mixed state is stabilized by LG . In fact this is the *unique* state that has this stabilizer. One can see this as follows.

Since \mathbb{C}^2 is an irreducible representation of $SU(2)$, it follows that $(\mathbb{C}^2)^{\otimes n}$ is an irreducible representation of $G = (SU(2))^n$ (see [19], Proposition 4.14). By Schur's lemma, any matrix that commutes with every $g \in G$ must be a scalar matrix ([19], Theorem 1.10).

Alternatively, it is straightforward to verify that if ρ has an off-diagonal element ρ_{IJ} with $I \neq J$, with say $i_k \neq j_k$, then $\langle I | [A^{(k)}, \rho] | J \rangle = (-1)^{i_k} 2i\rho_{IJ} \neq 0$ (use (8)). This rules out nonzero off-diagonal elements. Now suppose $\rho_{II} \neq \rho_{I_k I_k}$ for some k , where I_k is the same as I in all positions but opposite in position k . Then $\langle I | [C^{(k)}, \rho] | J \rangle = i(\rho_{I_k I_k} - \rho_{II}) \neq 0$ (use (9)). We conclude that ρ is the completely mixed state.

VI. STABILIZER IS K_{Werner}

Let ρ be an n -qubit symmetric mixed state, and suppose that $K_\rho = K_{\text{Werner}}$. It follows that, on the group level, we have $\text{Stab}_\rho \supset \{(g, g, \dots, g) : g \in SU(2)\}$. This is the class of Werner states. In [8] we prove that any symmetric Werner state must be of the form

$$\rho = \sum_{k=0}^{\lfloor n/2 \rfloor} c_k \text{Sym} \left(u^{\otimes k} \otimes \text{Id}^{\otimes n-2k} \right)$$

where

$$u = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$$

for some real coefficients $c_{k,\ell}$. Further, we show that any two states with distinct choices of coefficients $c_{k,\ell}$ are local unitarily inequivalent.

VII. STABILIZER IS K_{product}

It is clear that if $\rho = \sum_I s_I \sigma_I$ where $s_I = 0$ if $i_k = 1, 2$ for every $1 \leq k \leq n$, then K_ρ contains K_{product} . As long as there is some $s_I \neq 0$ for $I \neq (0, 0, \dots, 0)$, we can apply (9) to see that $[C^{(k)}, \rho] = 0$, so we have $K_\rho = K_{\text{product}}$ (the only possible algebra that contains K_{product} is all of LG).

Conversely, suppose $K_\rho = K_{\text{product}}$. Write ρ as

$$\begin{aligned} \rho &= \sum_J a_J \sigma_0 \otimes \sigma_J + \sum_J b_J \sigma_1 \otimes \sigma_J \\ &+ \sum_J c_J \sigma_2 \otimes \sigma_J + \sum_J d_J \sigma_3 \otimes \sigma_J \end{aligned}$$

where J is a multiindex of length $n-1$. Then we have

$$0 = [A_1, \rho] = \sum_J (-2b_J \sigma_3 + 2c_J \sigma_2) \otimes \sigma_J.$$

It follows that $b_J = c_J = 0$ for all J . Thus we have established that $K_\rho = K_{\text{product}}$ if and only if

$$\rho = \sum_{k=0}^n c_k \text{Sym} \left(\sigma_z^{\otimes k} \otimes \text{Id}^{\otimes (n-k)} \right)$$

for some real coefficients c_k .

In contrast to the case of symmetric Werner states in the previous section, LU equivalence classes of states whose stabilizer is K_{product} may contain more than one state. For example, let $\rho = \text{Id}/8 + a\sigma_z \otimes \sigma_z \otimes \sigma_z$. Then $\rho' = (iX)^{\otimes 3} \rho (-iX)^{\otimes 3} = \text{Id}/8 - a\sigma_z \otimes \sigma_z \otimes \sigma_z$ is LU equivalent but not equal to ρ , and has the same stabilizer (here X denotes the Pauli matrix σ_x viewed as an element $U(2)$). We claim that this type of sign change accounts for the entire LU class of a state for $n \geq 3$ qubits. That is, if ρ, ρ' are LU equivalent $n \geq 3$ qubit states with stabilizer $K_\rho = K_{\rho'} = K_{\text{product}}$, and ρ has corresponding polynomial $F_n(\rho) = \sum_k d_k z^k$, then either $\rho = \rho'$ or $F_n(\rho') = \sum_k (-1)^k d_k z^k$. To see why this is true, suppose we have $g\rho g^\dagger = \rho'$ for some $g = (g_1, \dots, g_n)$ in $SU(2)^n$. We show in [3] that for $n \geq 3$, there exists an $h \in SU(2)$ such that $\rho' = T_h(\rho) = h^{\otimes n} \rho (h^\dagger)^{\otimes n}$. By Lemma 1 we have $hAh^\dagger = \pm A$. Thus we have $R_h(z) = \pm z$.

Lemma 1 admits one additional possibility for 2-qubit states, and that is to have one of g_1, g_2 be diagonal, and the other antidiagonal. An example is conjugation by $\text{Id} \otimes (iX)$ that takes $\rho = \frac{\text{Id}}{4} + a\sigma_3 \otimes \sigma_3$ to $\rho' = \frac{\text{Id}}{4} - a\sigma_3 \otimes \sigma_3$. It is easy to check that this exhausts all possibilities.

VIII. STABILIZER IS K_{GHZ}

First, we claim that if $K_\rho = K_{\text{GHZ}}$, then $\rho_{I,J} = 0$ for $J \neq I, I^c$, where I^c denotes the bitwise complement of I . Indeed, if $J \neq I, I^c$, then there exist qubit indices k, ℓ such that $i_k = j_k$ and $i_\ell = j_\ell^c$. Then we have (apply (8))

$$0 = \langle I | [A^{(k)} - A^{(\ell)}, \rho] | J \rangle = \zeta(I, J) = \pm 2i\rho_{I,J}.$$

Second, we claim that if $\rho_{I,I^c} \neq 0$, then $\{I, I^c\} = \{00 \dots 0, 11 \dots 1\}$. Simply note that for any pair of positions k, ℓ , we must have

$$0 = (-1)^{i_k} - (-1)^{i_\ell}$$

so $i_k = i_\ell$.

Let $|\psi(n, k)\rangle = |1\rangle^{\otimes k} \otimes |0\rangle^{\otimes (n-k)}$ and let $\rho(n, k) = \text{Sym} |\psi(n, k)\rangle \langle \psi(n, k)|$. Let $|G(\alpha, \beta)\rangle = \alpha |0\rangle^{\otimes n} + \beta |1\rangle^{\otimes n}$, and let $\rho_G(\alpha, \beta) = |G(\alpha, \beta)\rangle \langle G(\alpha, \beta)|$. The above two claims imply that if $K_\rho = K_{\text{GHZ}}$ then ρ can be written as a mixture of the form

$$\rho = \sum_{k=0}^n c_k \rho(n, k) + d \rho_G(\alpha, \beta) \quad (14)$$

where c_0, \dots, c_n, d are nonnegative real numbers that sum to 1.

Now suppose that ρ, ρ' are LU equivalent n -qubit states of the form (14) for some $n \geq 3$. As was the case in the previous section, there is some $g \in SU(2)$ such that $\rho' = T_g(\rho)$, and g is either diagonal or antidiagonal. If $g = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$, then conjugation by $g^{\otimes n}$ leaves $\rho(n, k)$ fixed and performs a phase operation on $\rho_G(\alpha, \beta)$ as follows.

$$\begin{aligned} T_g(\rho(n, k)) &= \rho(n, k) \\ T_g(\rho_G(\alpha, \beta)) &= \rho_G(e^{itn}\alpha, e^{-itn}\beta) \end{aligned}$$

If $g = \begin{bmatrix} 0 & e^{-it} \\ e^{it} & 0 \end{bmatrix}$, then T_g interchanges $\rho(n, k)$ with $\rho(n, n-k)$ and transforms $\rho_G(\alpha, \beta)$ as follows.

$$\begin{aligned} T_g(\rho(n, k)) &= \rho(n, n-k) \\ T_g(\rho_G(\alpha, \beta)) &= \rho_G(e^{-itn}\beta, e^{itn}\alpha) \end{aligned}$$

The considerations of the preceding paragraph imply that if we exercise LU freedom to choose α, β real with $\alpha \geq \beta$, then (14) is a unique representative of its LU class of states, with the only exception being the case when $\alpha = \beta$. In this case, $\rho = \sum_k c_k \rho(n, k) + d \rho_G(1/\sqrt{2}, 1/\sqrt{2})$ and $\rho' = \sum_k c'_k \rho(n, k) + d \rho_G(1/\sqrt{2}, 1/\sqrt{2})$ are interchanged by T_{iX} .

IX. STABILIZER IS K_{Dicke}

Let ρ be an n -qubit symmetric mixed state. We claim that $g^{\otimes n} \rho (g^\dagger)^{\otimes n} = \rho$ for all *diagonal* $g \in SU(2)$ if and only if $F_n(\rho)$ is a linear combination of products of z^r and $(x^2 + y^2)^s$, for $0 \leq r + 2s \leq n$.

As for the case of the single $su(2)$ block, we make use of the identification of symmetric states with polynomials given in section III.

If g is a diagonal element of $SU(2)$, then $\Phi(g)$ is a rotation about the z axis. Polynomials $F_n(\rho) = \sum_{r,s} b_{r,s} z^r (x^2 + y^2)^s$ (where the sum is over nonnegative integers r, s such that $r + 2s \leq n$) are invariant since z and $x^2 + y^2$ are invariant under rotations about the z axis.

Conversely, suppose that $g^{\otimes n} \rho (g^\dagger)^{\otimes n} = \rho$ for all diagonal $g \in SU(2)$. Then $F_n(\rho)$ is invariant under rotations about the z axis. We now wish to decompose the homogeneous polynomials in three variables into a sum of irreducible representations of $U(1)$, and to note the dimension of the trivial representation, as it gives the space of polynomials invariant under rotations about the z axis. Each V_l in (13) decomposes into $2l + 1$ one-dimensional irreducible representations of $U(1)$, of which one is the trivial representation. When expressed as a sum of irreducible representations of $U(1)$, decomposition (13) shows that the homogeneous polynomials of degree p in three variables contain $\lfloor p/2 \rfloor + 1$ dimensions of the trivial representation. This is precisely the dimension of the space of homogeneous polynomials of degree p that

can be produced by products of z^r and $(x^2 + y^2)^s$ (i.e. the number of nonnegative integer pairs (r, s) for which $r + 2s = p$). The vector space $\mathbb{R}_n[x, y, z]$ of polynomials of degree at most n is a direct sum of vector spaces of homogeneous polynomials of degree p , for $0 \leq p \leq n$. Since we have accounted for the dimensions of each space of homogeneous polynomials, $F_n(\rho)$ must be a linear combination of the polynomials given.

Now suppose that ρ, ρ' are LU equivalent, with $F_n(\rho) = \sum_{r,s} b_{r,s} z^r (x^2 + y^2)^s$, and $F_n(\rho') = \sum_{r,s} b'_{r,s} z^r (x^2 + y^2)^s$. For $n \geq 3$ qubits, the same analysis as for symmetric Werner states (Section VI) yields a diagonal or antidiagonal $g \in SU(2)$ such that $T_g(\rho) = \rho'$. Since $R_g(x^2 + y^2) = x^2 + y^2$ and $R_g(z) = \pm z$, the only nontrivial possibility is that $F_n(\rho') = \sum_{r,s} (-1)^r b_{r,s} z^r (x^2 + y^2)^s$. We can discard the case $n = 2$, because the only possibilities for ρ are $\rho = \text{Id}/4 + a(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)$ and $\rho = \text{Id}/4 + a\sigma_z \otimes \sigma_z$. The first of these states has an $su(2)$ block stabilizer, and the second has stabilizer K_{product} .

X. SUMMARY AND CONCLUSION

Table I shows a summary of LU classes of local unitary stabilizer algebra types and their corresponding LU classes of states. Having achieved LU classification for symmetric mixed states, it is natural to attempt further classes of mixed states. A natural avenue for investigation is to take the stabilizer structures from Theorem 2 and try to classify their corresponding states, with the assumption of permutation invariance removed. For example, the case Werner states (stabilizer type (b)) would be of interest.

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Appendix A: Proofs for Section IV

1. Proof of Theorem 1

Let K be a subalgebra of $LG = \bigoplus_{i=1}^n su(2)$. For each qubit i , let $\pi_i: LG \rightarrow su(2)$ denote the projection $(M_1, M_2, \dots, M_i, \dots, M_n) \rightarrow M_i$ onto the i th direct summand of LG . Given a set \mathcal{S} of qubits, let $\pi_{\mathcal{S}}$ denote the projection $\pi_{\mathcal{S}} = \bigoplus_{s \in \mathcal{S}} \pi_s$ onto summands in qubits \mathcal{S} . Given $M = (M_1, \dots, M_n)$ in LG , let the *weight* of M , denoted $\text{wt}(M)$, be the number of i such that M_i is not zero.

For each $i \in \{1, 2, \dots, n\}$, let $m(i)$ be the minimum weight of all elements of K with nonzero weight in position i . For i such that $m(i) > 0$, let $M(i) \subseteq K$ denote

the set of elements of K that have nonzero projection in position i and have total weight $m(i)$. That is,

$$\begin{aligned} m(i) &= \min\{\text{wt}(M) : M \in K : M_i \neq 0\}, \text{ and} \\ M(i) &= \{M \in K : M_i \neq 0 \text{ and } \text{wt}(M) = m(i)\}. \end{aligned}$$

Proposition 1. *Suppose $\dim \pi_i(K) = 3$, and let P, Q be elements of $M(i)$. For $1 \leq j \leq n$ we have $P_j = 0$ if and only if $Q_j = 0$.*

Proof. Let U' be an element of K such that U'_i, P_i are independent elements of $su(2)$. Let $V = P/|P|_i$, let $W = [U', V]/|[U', V]_i|$, and let $U = [V, W]$.

Observation 1. We have that U_i, V_i, W_i form an orthonormal [23] basis of $su(2)$ that satisfy $[U_i, V_i] = W_i$, $[V_i, W_i] = U_i$, and $[W_i, U_i] = V_i$. This is easy to see using the fact that $su(2)$ is isomorphic to \mathbb{R}^3 , with the Lie bracket operation corresponding to the cross product, and the (rescaled) Hilbert-Schmidt norm corresponding to the standard norm on \mathbb{R}^3 .

Observation 2. We have that U, V, W are elements of $M(i)$. This follows from the observation that in K , we have $\text{wt}([M, N]) \leq \text{wt}(M)$ for all M, N , so U, V, W all have weight less than or equal to $\text{wt}(P)$, and hence must have weight equal to $\text{wt}(P) = m(i)$. Further, we see that U, V, W must have zero and nonzero coordinates in the same positions as P .

By Observation 1, we have that Q_i is not a scalar multiple of at least one of U_i, V_i, W_i . Without loss of generality, suppose Q_i, U_i are linearly independent. Let $S = [Q, U]$, so by construction, we have $S_i \neq 0$, $S \in M(i)$, and S has zero and nonzero coordinates in the same positions as U , and hence in the same positions as P . If $Q_j \neq 0$, then S_j must be nonzero since $\text{wt}(S) = \text{wt}(Q) = m(i)$ and S has the same zero and nonzero coordinate positions as P , which also has weight $m(i)$. Thus $U_j \neq 0$, and therefore $P_j \neq 0$. Conversely, if $P_j \neq 0$, then $S_j \neq 0$, and therefore $Q_j \neq 0$. This concludes the proof. \square

Proposition 2. *Suppose $\dim \pi_i(K) = 3$. Let $\mathcal{B} \subseteq \{1, 2, \dots, n\}$ be the set of positions where all the elements of $M(i)$ have their nonzero coordinates, by virtue of Proposition 1. Then the following hold.*

- (i) *If $P \in K$ and $P_j = 0$ for all $j \notin \mathcal{B}$, then $P \in M(i)$ or $P = 0$.*
- (ii) *Any such P lies in the linear span of U, V, W constructed in the proof of Proposition 1.*
- (iii) *The elements U, V, W satisfy $[U, V] = W$, $[V, W] = U$, $[W, U] = V$ so that the linear span of U, V, W is isomorphic to $su(2)$.*
- (iv) *$K = \pi_{\mathcal{B}}(K) \oplus \pi_{\mathcal{B}^c}(K)$.*
- (v) *$\pi_{\mathcal{B}}(K)$ is an $su(2)$ block algebra.*

Stabilizer	LU Representative	LU Nonuniqueness	Pure State Example
LG	$\text{Id}/2^n$	unique	none
K_{Werner}	$\sum_{k=0}^{\lfloor n/2 \rfloor} c_k \text{Sym} \left((\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)^{\otimes k} \otimes \text{Id}^{\otimes (n-2k)} \right)$ c_k real, at least one $c_k \neq 0$ with $k > 0$	unique	singlet
K_{product}	$\sum_{k=0}^n c_k \text{Sym} \left(\sigma_z^{\otimes k} \otimes \text{Id}^{\otimes (n-k)} \right)$ c_k real, at least one c_k nonzero with $k > 0$	$c'_k = (-1)^k c_k$	product state
K_{GHZ}	$\sum_{k=0}^n c_k \rho(n, k) + d \rho_G G(\alpha, \beta)$ $\alpha \geq \beta > 0$	$c'_k = c_{n-k}$	GHZ
K_{Dicke}	$\sum_{2r+s \leq n} b_{r,s} \text{Sym} \left((\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)^{\otimes s} \otimes \sigma_z^{\otimes r} \otimes \text{Id}^{\otimes (n-r-2s)} \right)$ $b_{r,s}$ real, at least one $b_{r,s} \neq 0$ with $r > 0$	$b'_{r,s} = (-1)^r b_{r,s}$	Dicke state

TABLE I: Summary of LU classes of stabilizers and states. Each of the three last stabilizer types has precisely two LU inequivalent states of the form given in the column ‘LU Representative’. The column ‘LU Nonuniqueness’ gives conditions on the coefficients for the alternative state.

Proof. (i). If $P = 0$ or if $P_i \neq 0$ we are done, so suppose $P_i = 0$, and let $j \neq i$ be an element of \mathcal{B} such that $P_j \neq 0$. Let U, V, W be constructed as in the proof of Proposition 1. Since U_j, V_j, W_j are nonzero, they are independent elements of $su(2)$. Write $P_j = aU_j + bV_j + cW_j$, and let $Q = P - aU - bV - cW$. Then $Q_i = -aU_i - bV_i - cW_i \neq 0$, so $Q \in M(i)$. But $Q_j = 0$, and this contradicts Proposition 1. We conclude that $P_i = 0$ is impossible, so $P \in M(i)$.

(ii). Using (i), for $P \neq 0$ we can write $P_i = aU_i + bV_i + cW_i$ and let $Q = P - aU - bV - cW$. Then $Q_i = 0$, so $Q = 0$, again by (i). Thus $P = aU + bV + cW$.

(iii). We only need to show that $[U_k, V_k] = W_k$, $[V_k, W_k] = U_k$, and $[W_k, U_k] = V_k$ for $k \in \mathcal{B}$. By construction, we have $[V, W] = U$, and therefore we have $[V_k, W_k] = U_k$. We also have $[W_k, U_k] = \alpha_k V_k$ and $[U_k, V_k] = \beta_k W_k$ for some scalars α_k, β_k . Let $N = [V, [U, V]]$, and let $M = N - U$. Then $N_k = [V_k, \beta_k W_k] = \beta_k U_k$. Since $M_i = 0$, we must have $M_k = (\beta_k - 1)U_k = 0$ by (i), so we must have $\beta_k = 1$ for all $k \in \mathcal{B}$. A similar argument shows that $\alpha_k = 1$ for all $k \in \mathcal{B}$.

(iv). Let $M \in A$, and write $M = P + N$, where $P = \pi_{\mathcal{B}}(M)$ and $N = M - P$. We will show that $P \in A$. We have $[M, X] = [P, X]$ for $X = U, V, W$, so we can use (ii) to write $[P, X]$ as a linear combination of U, V, W .

$$\begin{aligned} [P, U] &= a_U U + b_U V + c_U W \\ [P, V] &= a_V U + b_V V + c_V W \\ [P, W] &= a_W U + b_W V + c_W W \end{aligned}$$

For $k \in \mathcal{B}$, we can write $P_k = a_k U_k + b_k V_k + c_k W_k$. Using the relations (iii), we have

$$\begin{aligned} [P_k, U_k] &= -b_k W_k + c_k V_k \\ [P_k, V_k] &= a_k W_k - c_k U_k \\ [P_k, W_k] &= -a_k V_k + b_k U_k. \end{aligned}$$

Equating k th coefficients of the first set of equations with

coefficients in the second set of equations yields

$$\begin{aligned} a_k &= -b_W = c_V \\ b_k &= -c_U = a_W \\ c_k &= -a_V = b_U. \end{aligned}$$

Since this holds for all k , we have $P = c_V U + a_W V + b_U W$, and so $P \in K$, as desired.

(v). Let $b \in \mathcal{B}$. Because U_b, V_b, W_b satisfy $[U_b, V_b] = W_b$, $[V_b, W_b] = U_b$, and $[W_b, U_b] = V_b$, it must be that U_b, V_b, W_b form an orthonormal basis for $su(2)$ (again use the fact that $su(2)$ with its Lie bracket and Hilbert-Schmidt norm is isometrically identified with \mathbb{R}^3 with the cross product and standard euclidean norm). Because the adjoint representation $SU(2) \rightarrow SO(su(2))$ is surjective, we may choose $h_b \in SU(2)$ such that $h_b U_b h_b^\dagger = A$. Now choose a real number t_b such that $e^{it_b} V_b e^{-it_b} = B$. Now let $g_b = e^{it_b} h_b$. For $b' \notin \mathcal{B}$, let $g_{b'} = \text{Id}$. We now have

$$\begin{aligned} (\otimes_i g_i) U (\otimes_i g_i)^\dagger &= (A, A, \dots, A) \\ (\otimes_i g_i) V (\otimes_i g_i)^\dagger &= (B, B, \dots, B) \\ (\otimes_i g_i) W (\otimes_i g_i)^\dagger &= (C, C, \dots, C) \end{aligned}$$

and $\pi_{\mathcal{B}}(K)$ has been ‘‘aligned’’ with $\Delta_{\mathcal{B}}$, as desired. \square

To complete the proof of Theorem 1, for each qubit label i for which $\dim \pi_i(K) = 3$, let $\mathcal{B} \subseteq \{1, 2, \dots, n\}$ be the set of positions where all the elements of $M(i)$ have their nonzero coordinates, as in Propositions 1 and 2. Then $\pi_{\mathcal{B}}(K)$ is an $su(2)$ block summand for K by part (v) of Proposition 2. For any qubit j outside of the qubit sets for $su(2)$ blocks, we must have $\dim \pi_j(K) = 0, 1$, for if there are two independent vectors in $\pi_j(K)$, then there must also be a third coming from the bracket of the two independent elements, and so j would be a qubit for an $su(2)$ block. We define \mathcal{S} to be the set of qubits j for which $\dim \pi_j(K) = 1$ and define \mathcal{R} to be the remaining

qubits ℓ , where we have $\dim \pi_\ell(K) = 0$. With these definitions, we clearly have the desired decomposition of Theorem 1.

2. Proof of Theorem 2

Let ρ be a symmetric mixed state, and let K_ρ be its local unitary stabilizer subalgebra. By Theorem 1, we can decompose K_ρ as a direct sum

$$K_\rho = \left(\bigoplus_{i=1}^p B_i \right) \oplus S \quad (\text{A1})$$

of $su(2)$ blocks B_i and an algebra S where projections into each qubit summand are 1-dimensional.

To begin, note that permutation invariance implies that the dimension of the projection of K_ρ into any $su(2)$ summand of LG must be the same for all qubits. Thus we can have only the zero algebra (possibility (f) of Theorem 2), an algebra S , or a sum of $su(2)$ blocks.

Next, we claim that there are only two possibilities for a sum of $su(2)$ blocks. One extreme is to have n $su(2)$ blocks, each in 1 qubit (possibility (a) of Theorem 2). The other is to have a single $su(2)$ block in all n qubits. We rule out the intermediate possibilities, that is, having two or more $su(2)$ blocks, at least one of which involves two or more qubits, as follows. Suppose \mathcal{B}_1 contains qubits i and j , with corresponding $su(2)$ block algebra B_1 , and \mathcal{B}_2 contains qubit $k \neq i, j$, with corresponding $su(2)$ block algebra B_2 . Transposing qubits j, k does not affect ρ , so K_ρ also has an $su(2)$ block algebra B_3 that contains qubits i, k . But then the qubit sets for B_1 and B_3 both contain i . This contradicts the fact that qubit sets for $su(2)$ blocks are disjoint, as shown in Proposition 2.

To complete part (b) of Theorem 2, we consider two cases.

Part (b) case (i) $n \geq 3$. We claim that if K_ρ is a single $su(2)$ block in all $n \geq 3$ qubits, then in fact K_ρ is the standard $su(2)$ block algebra $\Delta_{\{1,2,\dots,n\}}$. Suppose on the contrary that there is an element $M = \sum_i M_i^{(i)}$ in K_ρ with $M_i \neq M_j$. Let $M' = M_j^{(i)} + M_i^{(j)} + \sum_{k \neq i,j} M_k^{(k)}$ be the element obtained from M by transposing the i, j coordinates. Then $M - M' = (M_i - M_j)^{(i)} + (M_j - M_i)^{(j)}$ is also in K_ρ , but this element has weight $2 < n$. This contradicts Proposition 2 that says all elements in an $su(2)$ block must have full weight n . We conclude that a single $su(2)$ block stabilizer for a symmetric mixed state of $n \geq 3$ qubits must be the standard $su(2)$ block algebra.

Part (b) case (ii) $n = 2$. We claim that any two-qubit symmetric mixed state with an $su(2)$ -block stabilizer is LU equivalent to another symmetric state with a standard $su(2)$ -block stabilizer. This LU equivalence may not be achievable through the same unitary operation on each qubit. For example, the pure state $|01\rangle + |10\rangle$ is a symmetric state with a non-standard $su(2)$ -block stabilizer. It is LU equivalent to the singlet state $|01\rangle - |10\rangle$,

but not through an LU transformation applied uniformly to each qubit. In fact, the singlet is invariant under any LU transformation applied uniformly to each qubit.

Suppose ρ is a 2-qubit symmetric mixed state whose stabilizer subalgebra is an $su(2)$ block subalgebra. Writing

$$\rho = \sum_{i,j=0}^3 s_{ij} \sigma_i \otimes \sigma_j,$$

we see that

$$s_{10} = s_{20} = s_{30} = s_{01} = s_{02} = s_{03} = 0$$

because the three components (s_{10}, s_{20}, s_{30}) transform like a 3-dimensional real vector under the rotations of 3-dimensional space produced by the first qubit in the $su(2)$ block subalgebra. Since ρ is invariant under the action of this $su(2)$ block subalgebra, it must be that $(s_{10}, s_{20}, s_{30}) = 0$. A similar argument holds for the second qubit. By performing singular value decomposition on the real 3×3 matrix of elements $s_{i_1 i_2}$ with i_1 and i_2 ranging over $\{1, 2, 3\}$, we can find a local unitary transformation such that $\rho' = (g_1 \otimes g_2) \rho (g_1 \otimes g_2)^\dagger$ has the form

$$\rho' = \frac{1}{4} (\sigma_0 \otimes \sigma_0 + a \sigma_1 \otimes \sigma_1 + b \sigma_2 \otimes \sigma_2 + c \sigma_3 \otimes \sigma_3).$$

Let

$$M = \alpha_1 A^{(1)} + \beta_1 B^{(1)} + \gamma_1 C^{(1)} + \alpha_2 A^{(2)} + \beta_2 B^{(2)} + \gamma_2 C^{(2)}.$$

This M stabilizes ρ' if and only if

$$\begin{aligned} \alpha_1 c &= \alpha_2 b \\ \alpha_1 b &= \alpha_2 c \\ \beta_1 a &= \beta_2 c \\ \beta_1 c &= \beta_2 a \\ \gamma_1 b &= \gamma_2 a \\ \gamma_1 a &= \gamma_2 b. \end{aligned}$$

Since there is, by assumption, an element M with $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2$, and γ_2 all nonzero, we must have

$$|a| = |b| = |c|.$$

If a, b , and c are not identical, then two have the same sign and the last has the opposite sign, and we can do a unitary transformation on one qubit that produces a π rotation about the x, y , or z axis, resulting in an LU equivalent state that is symmetric with the standard $su(2)$ block subalgebra as its stabilizer. This completes part (b), case (ii).

Now we suppose that $K_\rho = S$, so that the projection of K_ρ into each LG summand $su(2)$ is 1-dimensional. We wish to show that one of possibilities (c)–(e) of the statement of the Theorem holds. We do this by cases.

If K_ρ contains a weight one element $M^{(k)}$, then by permutation invariance, K_ρ must contain $M^{(j)}$ for $1 \leq k \leq n$. This establishes possibility (c).

Now suppose that K_ρ contains no weight one elements. There are two cases: there may exist a nonzero element $M = (M_1, M_2, \dots, M_n) = \sum_k M_k^{(k)}$ in K_ρ such that $M_i \neq M_j$ for some i, j , or there may not. Suppose the first case holds. For any pair i, j of qubit positions, we have the element $M' = M_j^{(i)} + M_i^{(j)} + \sum_{k \neq i, j} M_k^{(k)}$, obtained by transposing coordinates i, j , also in K_ρ . Therefore we have $M - M' = (M_i - M_j)^{(i)} + (M_j - M_i)^{(j)}$ also in K_ρ . Let $N = M_i - M_j$. We have $n - 1$ linearly independent weight two elements $N^{(1)} - N^{(k)}$, $2 \leq k \leq n$, so the dimension of K_ρ is at least $n - 1$. If the dimension of K_ρ is greater than $n - 1$, then there is an element in K_ρ of the form $M'' = \sum_{k=1}^n a_k N^{(k)}$ that does not lie in the span of the elements $N^{(i)} - N^{(j)}$. But then we would have the weight one element $(\sum_{k=1}^n a_k) N^{(k)} = M'' + \sum_{k=2}^n a_i (N^{(1)} - N^{(k)})$ in K_ρ , which contradicts our assumption. We conclude that K_ρ is the real span of $\{N^{(1)} - N^{(i)}\}_{i=2}^n$. We may take N to have norm 1, and we may choose $g \in SU(2)$ such that $gNg^\dagger = A$. Let $\rho' = g^{\otimes n} \rho (g^{\otimes n})^\dagger$. Then $K_{\rho'}$ is the real span of $\{A^{(i)} - A^{(j)}\}_{1 \leq i, j \leq n}$, which is the stabilizer (d) of Theo-

rem 2.

The last case to consider is where $M_i = M_j$ for all i, j , for all $M = (M_1, M_2, \dots, M_n)$ in $K_\rho = S$. Let $M = (N, \dots, N)$ be a nonzero element in K_ρ . Normalize N and choose $g \in SU(2)$ to diagonalize N so we have $gNg^\dagger = A$. Then the state $\rho' = g^{\otimes n} \rho (g^{\otimes n})^\dagger$ has 1-dimensional stabilizer that is the real span of $A^{(1)} + A^{(2)} + \dots + A^{(n)}$. This is type (e) in Theorem 2.

This concludes the proof of Theorem 2.

3. Proof of Lemma 1

Suppose that K_ρ is a stabilizer of type (c), (d), or (e), in Theorem 2, and that

$$(g_1 \otimes g_2 \otimes \dots \otimes g_n) K_\rho (g_1 \otimes g_2 \otimes \dots \otimes g_n)^\dagger = K_\rho$$

for some g_1, \dots, g_n in $SU(2)$. For M in any of the three stabilizer types, the assumption that $(\otimes_k g_k) M (\otimes_k g_k)^\dagger \in K_\rho$ implies that $g_k A g_k^\dagger = \pm A$ for each k . It is easy to check directly that if $g = \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix}$ satisfies $g A g^\dagger = \pm A$, then either a or b must be zero. Thus we conclude that g is either diagonal or antidiagonal, as desired.

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 - [20] When we identify the element $a\sigma_x + b\sigma_y + c\sigma_z$ in \mathcal{W} with $i(a\sigma_x + b\sigma_y + c\sigma_z)$ in $su(2)$, equation (1) is the adjoint action of $SU(2)$ on its Lie algebra $su(2)$, plus a trivial action on the real linear span of σ_0 . Equation (2) is the corresponding adjoint action of the Lie algebra $su(2)$ on itself, plus again the trivial action on σ_0 . Equations (3) and (4) are the natural extensions of (1) and (2) to tensor products. See, for example, [19].
 - [21] Φ is given in a natural way by the adjoint action $SU(2) \rightarrow SO(su(2))$, so that $\Phi(g)(M) = g M g^\dagger$, and we identify $su(2)$ with \mathbb{R}^3 by $A \leftrightarrow (1, 0, 0)$, $B \leftrightarrow (0, 1, 0)$, $C \leftrightarrow (0, 0, 1)$. See [19].
 - [22] To be precise, the cited work considers the representation $\mathbb{C}[x, y, z]$. In the case of $SO(3)$, the irreducible submodules of this complex representation are in one-to-one correspondence with the *real* irreducible submodules of $\mathbb{R}[x, y, z]$ via complexification. See [19, Ch.2 Sec.6].
 - [23] We use a rescaled Hilbert-Schmidt norm $|M| =$

$(2 \operatorname{tr}(M^\dagger M))^{1/2}$ for an element M of $su(2)$.